MATH 2020 Tutorial

① Solve the system

$$u = x - y, \quad v = 2x + y$$

for x, y in terms of u and v.

(a) Then find the value of the Jacobian $\partial(x,y)/\partial(u,v)$

(b) Find the image under the transformation $u = x-y$, $v = 2x+y$ of the trianglar region with vertices $(0,0)$, $(1,1)$, $(1,-2)$ in the xy plane.

Ans: (a) $\begin{cases} u = x - y \\ v = 2x + y \end{cases} \Rightarrow \begin{cases} x = \frac{u+v}{3} \\ y = \frac{v-2u}{3} \end{cases}$ $\left(y = x - u = \frac{u+v}{3} - u = \frac{v-2u}{3} \right).$

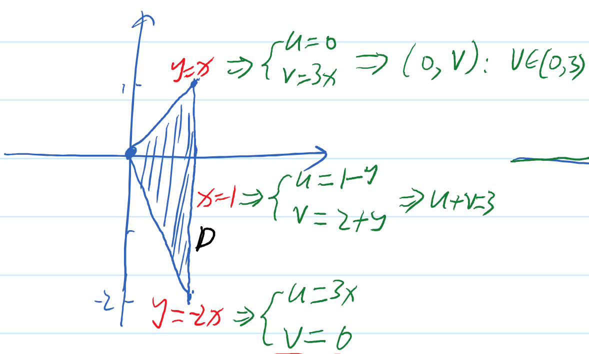
$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| = \left| \begin{matrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{matrix} \right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}.$$

Another method:

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \left| \begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix} \right| = 3$$

$$\therefore \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} = \frac{1}{3}$$

(b) $\begin{cases} u = x - y \\ v = 2x + y \end{cases}$



$$\int_D dxdy = \int_{D'} \frac{\partial(x,y)}{\partial(u,v)} dudv$$

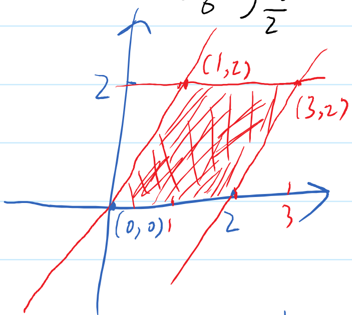
$$= \int_0^3 \int_0^{3-u} \frac{1}{3} \, dv \, du$$

② Use the transformation $\boxed{x = u + \frac{v}{2}, \ y = v}$ to evalute the integral

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} y^3 (2x-y) e^{(2x-y)^2} dxdy$$

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

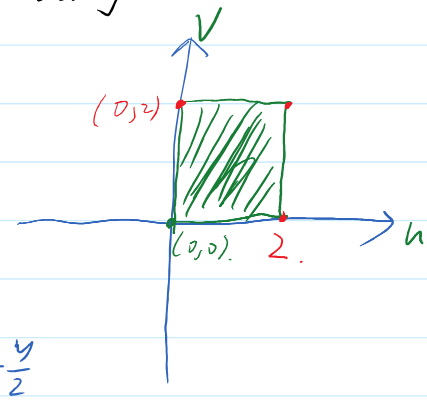
Ans:



$x=\frac{y}{2}$

$x=\frac{y}{2}+2.$

$y=2x-4.$



$$\begin{cases} x = u+\frac{v}{2} \\ y=v \end{cases} \Rightarrow \begin{cases} u = x-\frac{v}{2} = x-\frac{y}{2} \\ v=y \end{cases}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$$

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

$$= \int_0^2 \int_0^2 v^3\cdot(2u)e^{4u^2}\,du\,dv$$

$$= \int_0^2 \frac{v^3}{4} e^{4u^2}\Big|_0^2\,dv$$

$$= \int_0^2 \frac{v^3}{4}(e^{16}-1)\,dv$$

$$= \frac{v^4}{16}(e^{16}-1)\Big|_0^2 = e^{16}-1 \qquad \#.$$

③ Let D be the region in xyz-space defined by the inequalities

$1 \le x \le 2, \quad 0 \le xy \le 2, \quad 0 \le z \le 1.$

Evaluate
$$\iiint_D (x^2y+3xyz)\,dx\,dy\,dz$$

by applying the transformation

$u=x, \quad v=xy, \quad w=3z.$

Ans: $1 \le u \le 2, \quad 0 \le v \le 2, \quad 0 \le w \le 3$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{\begin{vmatrix} 1 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 3 \end{vmatrix}} = \frac{1}{3x} = \frac{1}{3u}.$$

$$\iiint (x^2y+3xyz)\,dx\,dy\,dz.$$

$$\iiint_D (x^2y+3xyz)\,dx\,dy\,dz.$$

$$= \int_1^2\int_0^2\int_0^3 (uv+vw)\ \frac{1}{3u}\,dw\,dv\,du$$

$$= \int_1^2\int_0^2\int_0^3 \frac{v}{3}+\frac{vw}{3u}\,dw\,dv\,du$$

$$= \int_1^2\int_0^2 \frac{v}{3}w+\frac{v}{6u}w^2\Big|_0^3\,dv\,du$$

$$= \int_1^2\int_0^2 v+\frac{3v}{2u}\,dv\,du$$

$$= \int_1^2 \left(1+\frac{3}{2u}\right)\frac{1}{2}v^2\Big|_0^2\,du$$

$$= \int_1^2 2+\frac{3}{u}\,du$$

$$= 2u+3\ln u\Big|_1^2 = (4+3\ln 2)-(2+0) = 2+3\ln 2.$$

④ (a) Evalute $I_1=\int_0^\infty e^{-xy}\sin y\,dy$, $I_2=\int_0^\infty e^{-xy}\cos y\,dy$

where x is a positive constant.

Ans: (a) $I_1=\int_0^\infty e^{-xy}\sin y\,dy = -\int_0^\infty e^{-xy}\,d\cos y$

$$= -e^{-xy}\cos y\Big|_0^\infty + \int_0^\infty -xe^{-xy}\cos y\,dy$$

$$= 0+1-x\int_0^\infty e^{-xy}\cos y\,dy = 1-x\cdot I_2.$$

$$I_2=\int_0^\infty e^{-xy}\cos y\,dy = \int_0^\infty e^{-xy}\,d\sin y$$

$$= e^{-xy}\sin y\Big|_0^\infty - \int_0^\infty -x\cdot e^{-xy}\sin y\,dy = x\cdot\int_0^\infty e^{-xy}\sin y\,dy$$

$$= x\cdot I_1$$

$$\begin{cases} I_1 = 1-x\cdot I_2 \\ \ldots \end{cases} \Rightarrow \begin{cases} I_1=\frac{1}{x^2+1}, & I_2=\frac{x}{x^2+1} \end{cases}$$

$$\begin{cases} I_1 = 1 - x\cdot I_2 \\ I_2 = x\cdot I_1 \end{cases} \Rightarrow \begin{cases} I_1 = \frac{1}{x^2+1}, & I_2 = \frac{x}{x^2+1} \end{cases}$$

(b) Show that $\int_0^\infty \frac{x\sin x}{x^2+1}\,dx = \int_0^\infty \frac{\cos x}{x^2+1}\,dx$.

Pf: $$\int_0^\infty \frac{x\sin x}{x^2+1}\,dx = \int_0^\infty \underbrace{\int_0^\infty e^{-xy}\cos y\,dy}_{I_2}\,\sin x\,dx$$

$$= \int_0^\infty \int_0^\infty e^{-xy}\sin x\,dx\,\cos y\,dy$$

$$= \int_0^\infty I_1 \cos y\,dy = \int_0^\infty \frac{\cos y}{y^2+1}\,dy.$$

(c) Evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$.

$$\frac{1}{x} = \int_0^\infty e^{-xy}\,dy \quad \left(\int_0^\infty e^{-xy}\,dy = -\frac{1}{x}e^{-xy}\Big|_0^\infty = \frac{1}{x} \right).$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \int_0^\infty e^{-xy}\,dy\,\sin x\,dx.$$

$$= \int_0^\infty \int_0^\infty e^{-xy}\sin x\,dx\,dy$$

$$= \int_0^\infty \frac{1}{y^2+1}\,dy$$

$$= \arctan y\Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

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of the trianglar region with vertices $(0,0)$, $(1,1)$, $(1,-2)$ in the xy plane.

Ans: $\begin{cases} u = x-y \\ v = 2x+y \end{cases} \Rightarrow \begin{cases} x = \dfrac{u+v}{3} \\ y = x-u = \dfrac{u+v}{3} - u = \dfrac{v-2u}{3} \end{cases}$

(a) $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{matrix}\right| = \dfrac{1}{9} + \dfrac{2}{9} = \dfrac{1}{3}$

Another method:

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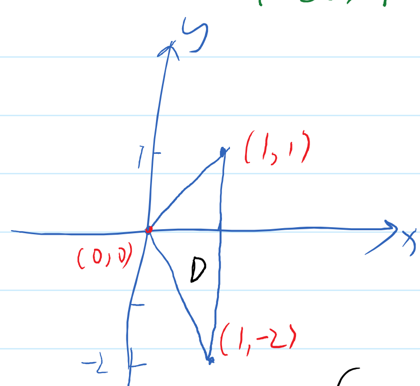
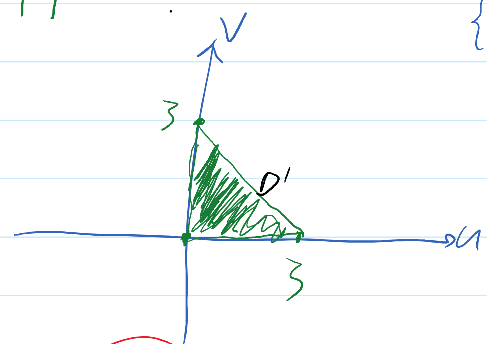
(b)





$\begin{cases} u = x-y \\ v = 2x+y \end{cases}$

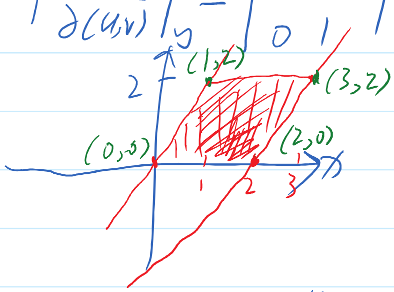
$$\int_D dxdy = \int_{D'} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$
$$= \int_{D'} \frac{1}{3}\, dudv.$$

② Use the transformation $x = u + \frac{v}{2}$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} y^3(2x-y)e^{(2x-y)^2}\, dxdy.$$

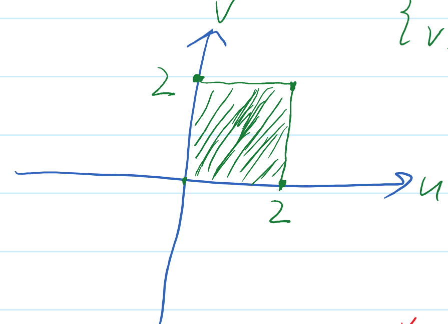
Ans: $\begin{cases} x = u + \frac{v}{2} \\ y = v \end{cases} \Rightarrow \begin{cases} u = x - \frac{v}{2} = x - \frac{y}{2} \\ v = y \end{cases}$

$\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} 1 & \frac{1}{2} \\ 0 & 1 \end{matrix}\right| = 1$



$x = \frac{y}{2} \Rightarrow y = 2x$

$x = \frac{y}{2} + 2 \Rightarrow y = 2x - 4$



$\begin{cases} u = x - \frac{y}{2} \\ v = y \end{cases}$

$$\int^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} y^3(2x-y)e^{(2x-y)^2}\, dxdy$$

$\begin{cases} x = u + \frac{v}{2} \\ y = v \end{cases}$

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

$$\begin{cases} x = u + \frac{y}{2} \\ y = v. \end{cases}$$

$$= \int_0^2 \int_0^2 v^3 \cdot 2u\, e^{4u^2} \quad 1 \cdot du\,dv.$$

$$= \int_0^2 v^3 \cdot \frac{1}{4} e^{4u^2} \Big|_0^2 \,dv$$

$$= \int_0^2 v^3 \cdot \frac{1}{4}(e^{16}-1)\,dv$$

$$= \frac{1}{4}v^4 \cdot \frac{1}{4}(e^{16}-1)\Big|_0^2$$

$$= e^{16}-1. \quad \#.$$

③ Let D be the region in xyz-space defined by the inequalities

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Evalute

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by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z.$$

$$1 \le u \le 2, \quad 0 \le v \le 2, \quad 0 \le w \le 3.$$

Ans:

$$\begin{cases} u = x \\ v = xy \\ w = 3z \end{cases} \Rightarrow \begin{cases} x = u \\ y = \frac{v}{x} = \frac{v}{u} \\ z = \frac{1}{3}w \end{cases}$$

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}.$$

$$= \frac{1}{\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{\begin{vmatrix} 1 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 3 \end{vmatrix}} = \frac{1}{3x} = \frac{1}{3u}.$$

$$\begin{cases} u = x \\ v = xy \\ w = 3z \end{cases}$$

$$\iiint_D (x^2y + 3xyz)\,dx\,dy\,dz.$$

$$= \int_1^2 \int_0^2 \int_0^3 (uv + vw) \quad \frac{1}{3u}\,dw\,dv\,du$$

$$= \int_1^2 \int_0^2 \int_0^3 \frac{v}{3} + \frac{vw}{3u}\,dw\,dv\,du$$

$$= \int_1^2\int_0^2\int_0^3 \frac{v}{3} + \frac{vw}{3u}\, dw\, dv\, du$$

$$= \int_1^2\int_0^2 \frac{v}{3}w + \frac{v}{6u}w^2 \Big|_0^3 dv\, du$$

$$= \int_1^2\int_0^2 v + \frac{3v}{2u}\, dv\, du$$

$$= \int_1^2 \frac{1}{2}v^2 + \frac{3}{4u}v^2 \Big|_0^2 du$$

$$= \int_1^2 2 + \frac{3}{u}\, du$$

$$= 2u + 3\ln u \Big|_1^2 = (4 + 3\ln 2) - (2+0) = 2 + 3\ln 2 \ \#$$

④ (a) Evalute $I_1 = \int_0^\infty e^{-xy} \sin y\, dy$, $I_2 = \int_0^\infty e^{-xy} \cos y\, dy$

where x is a positive constant.

Ans: $I_1 = \int_0^\infty e^{-xy} \sin y\, dy = -\int_0^\infty e^{-xy}\, d\cos y$

$$= -e^{-xy}\cos y \Big|_0^\infty + \int_0^\infty -xe^{-xy}\cos y\, dy$$

$$= 1 - x\int_0^\infty e^{-xy}\cos y\, dy = 1 - x I_2.$$

$$I_2 = \int_0^\infty e^{-xy}\cos y\, dy = \int_0^\infty e^{-xy}\, d\sin y$$

$$= e^{-xy}\sin y \Big|_0^\infty - \int_0^\infty -x e^{-xy}\sin y\, dy$$

$$= x\int_0^\infty e^{-xy}\sin y\, dy = x \cdot I_1$$

$$\begin{cases} I_1 = 1 - x\cdot I_2 \\ I_2 = x\cdot I_1 \end{cases} \Rightarrow \begin{cases} I_1 = \frac{1}{x^2+1} = \int_0^\infty e^{-xy}\sin y\, dy \\ I_2 = x\cdot I_1 = \frac{x}{x^2+1} = \int_0^\infty e^{-xy}\cos y\, dy \end{cases}$$

(b) Show that $\int_0^\infty \frac{x\sin x}{x^2+1}\, dx = \int_0^\infty \frac{\cos x}{x^2+1}\, dx$

$$\int_0 \frac{\quad}{x^2+1}\,dx = \int_0 \frac{\quad}{x^2+1}\,dx$$

Pf:
$$\int_0^\infty \frac{x}{x^2+1}\cdot \sin x\,dx = \int_0^\infty \int_0^\infty e^{-xy}\cos y\,dy\,\sin x\,dx.$$
$$= \int_0^\infty \int_0^\infty e^{-xy}\sin x\,dx\,\cos y\,dy \quad \text{[Fubini's Theorem]}$$
$$= \int_0^\infty \frac{1}{y^2+1}\cos y\,dy = \int_0^\infty \frac{\cos x}{1+x^2}\,dx.$$

(c) Evalute $\int_0^\infty \frac{\sin x}{x}\,dx$.

Ans:
$$\int_0^\infty e^{-xy}\,dy = -\frac{1}{x}e^{-xy}\Big|_0^\infty = 0-\left(-\frac{1}{x}e^0\right) = \frac{1}{x}.$$
$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \int_0^\infty e^{-xy}\,dy\cdot\sin x\,dx.$$
$$= \int_0^\infty \underbrace{\int_0^\infty e^{-xy}\sin x\,dx}_{I_1}\,dy$$
$$= \int_0^\infty \frac{1}{y^2+1}\,dy$$
$$= \arctan y\Big|_0^\infty = \frac{\pi}{2}-0 = \frac{\pi}{2} \quad \#.$$